# Boundedness of singularities & discreteness of local volume (j/w C. Xu)

## Boundedness of Fano varieties:

**Thm** (K-moduli thm) $\forall\ n = \dim$, $\varepsilon > 0$ (= volume)

$\exists$ proj. moduli sp parametrizing $\underbrace{\text{K-polystable}}_{\iff \exists \text{ Kähler-Einstein metric}}$ Fano var. of $\dim = n$ & volume $= (-K)^n \geq \varepsilon$.

## Boundedness part:

**Thm** (Jiang, Xu-Z) K-semistable Fano var. of $\dim = n$ & volume $\geq \varepsilon$ form a bounded set.

**Q**: local analog?

Fano variety $\longleftrightarrow$ klt singularities ($A_X(E) > 0$)

$V \longrightarrow \operatorname{Cone}(V, -K_V)$ klt

klt sing "=" (orbifold) cone / Fano + ($\mathbb{Q}$-Gorenstein) deform.

**Ex** $V$ = KE Fano mfd

$\operatorname{Cone}(V, -K_V)$ has a Ricci-flat Kähler cone metric

"K-ps sing" $\longleftrightarrow$ $\exists$ Ricci-flat Kähler cone metric

**Ex** Every toric klt sing has Ricci-flat Kähler cone metric (Futaki-Ono-G. Wang)

but not every toric Fano has KE (e.g. $(Bl_p \mathbb{P}^2)$)

## Stable degeneration of klt sing.

**Thm** (Blum, Li, Wang, Xu, – ; Collins-Székelyhidi, Li, Huang)

(1) Every klt sing. has a volume preserving isotrivial degeneration to a K-ps Fano cone singularity.

(2) Fano cone sing has a Ricci-flat Kähler cone metric $\iff$ it is K-ps

<u>Upshot</u>: $\forall$ klt sing is S-equiv. to a K-ps Fano cone sing.

<u>Rmk</u>
- local, alg. analog of Ricci flow.
- deform sp of non-isolated sing can be $\infty$-dim'l.

<u>Local volume</u> (Chi Li) $x \in X$ klt sing. $n = \dim X$

<u>Def</u>. (Li) normalized volume of valuations $v \in \mathrm{Val}_{X,x}$

$$\widehat{\mathrm{vol}}(v) = A_X(v)^n \cdot \mathrm{vol}(v)$$

<u>log discrepancy</u>:
- $v = \mathrm{ord}_E$, $E \subset Y \to X$

$$A_X(v) = A_X(E)$$

- $A_X(\lambda v) = \lambda A_X(v)$, $\lambda > 0$
- extend $A_X$ to general val. by lsc.

<u>Volume of valuation</u>: $\mathfrak{a}_\lambda(v) = \{ f \in \mathcal{O}_{X,x} \mid v(f) \geq \lambda \}$.

$$\mathrm{vol}(v) = \limsup_{m \to \infty} \frac{\ell(\mathcal{O}_{X,x}/\mathfrak{a}_m(v))}{m^n/n!}$$

<u>Local volume</u> $\widehat{\mathrm{vol}}(x, X) = \inf_v \widehat{\mathrm{vol}}(v)$.

<u>Ex</u> $0 \in \mathbb{A}^n$, $v = \mathrm{wt}_\alpha$ $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$

$$\widehat{\mathrm{vol}}(v) = \frac{(\alpha_1 + \dots + \alpha_n)^n}{\alpha_1 \cdots \alpha_n} \geq n^n$$

- <u>Li-Xu</u> $x \in X$ Ricci-flat Fano cone sing.

(under some "nice condition") $\widehat{\mathrm{vol}}(x, X) =$ volume density

$$= \lim_{r \to 0^+} \frac{\mathrm{Vol}(B_r(x, X))}{\mathrm{Vol}(B_r(0, \mathbb{C}^n))}$$

- <u>Li, Liu, Xu</u> $V =$ KE Fano var. $\dim = n-1$

$$X = \mathrm{Cone}(V, -K_V) \ni x = \text{vertex}$$

$\widehat{\mathrm{vol}}(x, X) = (-K_V)^{n-1}$ (Fix $n \in \mathbb{N}$, $\varepsilon > 0$)

<u>Thm</u> (Xu, -) K-semistable Fano cone sing. of dim $= n$, & $\widehat{\mathrm{vol}} \geq \varepsilon$ form a bounded set.

<u>Cor 1</u> $\widehat{\mathrm{Vol}}_n = \{$local volumes of $n$-dim klt sing$\}$ is discrete away from zero.

<u>Rmk</u> local volume can be irrat'l ($\mathrm{Cone}/\mathrm{Bl}_p\mathbb{P}^2$)

<u>Thm $\Rightarrow$ Cor 1.</u>
- <u>Stable degen. thm</u> $\Rightarrow \widehat{\mathrm{Vol}}_n = \{$loc. vol. of K-ss Fano cone$\}$
- $\widehat{\mathrm{vol}} \geq \varepsilon \geq 0 \xRightarrow{\text{Thm}}$ bdd.
- Xu: $\widehat{\mathrm{vol}}$ is constructible in family
  $\Downarrow$
  takes $< \infty$ many values. #

<u>Cor 2</u> K-ss Fano var. of dim $= n$ & (Weil index)$\cdot (-K)^n \geq \varepsilon > 0$ is bdd.

<u>Rmk</u> Weil index of $V = \max\{q \mid -K_V \sim_{\mathbb{Q}} qL,\ L \text{ Weil div}\} \geq 1$

<u>Cor 2</u> $\Rightarrow$ Global Bdd Thm (Jiang, XZ)

Weil index can be arbitrary large.

<u>Thm $\Rightarrow$ Cor 2</u> $X = \mathrm{Cone}(V, L)$, $\widehat{\mathrm{vol}}(x) = q \cdot (-K_V)^{n-1}$

$V$ K-ss $\Rightarrow X$ K-ss Fano cone sing.

<u>Known cases of thm</u>
- dim $= 2$, $X \cong \mathbb{C}^2/G$, $G \subset GL(2, \mathbb{C})$, $\widehat{\mathrm{vol}}(x) = \frac{4}{|G|}$
- dim $= 3$, Liu-Moraga-Süss, Z
- complexity $\leq 1$: LMS
- $(X, D)$, $X$ bounded: Han-Liu-Qi ("embedded version")

## Strategy of pf.

**Step 1.** Attach a global object (& reduce to global statement)

- Fano cone sing = klt sing w/ $T$-torus action

  vertex = fixed pt

- $X$ Fano cone sing = $\mathrm{Cone}(V, L)$

**Issue:** • $\exists$ many choice of $V$

(e.g. $X = \mathbb{A}^n$, $V = \mathbb{P}(\underline{a})$ )

• $\widehat{\mathrm{vol}}(X) \leftrightarrow \mathrm{vol}(V)$ is not directly related.

**Solution** $X \subset \overline{X}$ = projective orbifold cone / $(V, L)$

**Lem (Z)** $\overline{X}$ is Fano, $(-K_{\overline{X}})^n$ can be made arbitrarily closed to $\widehat{\mathrm{vol}}(x, X)$

$\overline{X} \setminus X \cong V$

**Thm** $\left[\begin{array}{l} (\overline{X}, V),\ -(K_{\overline{X}} + V) \text{ ample.}\quad, \mathrm{vol}(-(K_{\overline{X}} + V)) \geq \varepsilon > 0. \\ \alpha_x(\overline{X}, V) \geq \alpha_0 \text{ away from } V, \end{array}\right.$

$\Rightarrow$ $\overline{X} \setminus V$ form bdd set.